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LETTER TO THE EDITOR

Radial moments of folding integrals for some non-spherical distributions I

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Abstract. Radial moments of folding integrals for some non-spherical distributions that appear in problems relating to inelastic scattering off nuclei are evaluated in simple closed form in terms of the radial moments of the folded functions.

Satchler‡ (1972) attempted to compute radial moments of some folding integrals which are useful in problems occurring in inelastic scattering off nuclei. However, though special cases are computed in his paper, the general problem is not examined.

The problem is as follows. Given any (scalar) function $f(\boldsymbol{r})$ of the position vector $\boldsymbol{r}$, we can write its expansion in multipoles

$$f(\boldsymbol{r})=\sum_{lm} f_{lm}(r)\, Y_l^m(\Omega) \tag{1}$$

where Ω stands, as usual, for the two angles θ, ϕ. The above expansion can be inverted in the form

$$f_{lm}(r)=\int f(\boldsymbol{r})\, Y_l^{m*}(\Omega)\, \mathrm{d}\Omega \tag{2}$$

where $\mathrm{d}\Omega=\sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi$ and we have used the orthonormality condition

$$\int Y_l^m(\Omega)\, Y_{l'}^{m'*}(\Omega)\, \mathrm{d}\Omega=\delta_{ll'}\delta_{mm'}. \tag{3}$$

Next we consider a folding function $h(\boldsymbol{r})$ defined by

$$h(\boldsymbol{r})=\int f(\boldsymbol{r}_1)g(|\boldsymbol{r}-\boldsymbol{r}_1|)\, \mathrm{d}\boldsymbol{r}_1 \tag{4}$$

by folding a non-spherically symmetric (scalar) function $f(\boldsymbol{r})$ with a *spherically symmetric* function $g(\boldsymbol{r})$. We can also write the multipole expansion for $h(\boldsymbol{r})$ which defines the coefficients $h_{lm}(r)$.

Now defining the nth *radial moment*

$$J_n(f)=4\pi\int_0^\infty f(r)r^{n+2}\, \mathrm{d}r \tag{5}$$

of any function $f(r)$, the problem is to connect the nth radial moments of $h_{lm}\ (n\geq l)$ with those of f_{lm} and g.

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‡ We have followed his notation throughout.

Using equations (1), (2), (4) and (5), we can immediately write

$$J_{l+2k}(h_{lm}) = 4\pi \int \sum_{l_1 m_1} f_{l_1 m_1}(r_1) g(s) Y_{l_1}^{m_1}(\Omega_1) Y_l^{m*}(\Omega) r^{l+2k} \, \mathrm{d}\boldsymbol{r} \, \mathrm{d}\boldsymbol{r}_1 \tag{6}$$

where

$$\boldsymbol{s} = \boldsymbol{r} - \boldsymbol{r}_1. \tag{7}$$

From the character of the above equation, it is evident that only the term with $l_1 = l$, $m_1 = m$ contributes in equation (6). Thus

$$J_{l+2k}(h_{lm}) = 4\pi \int f_{lm}(r_1) g(s) Y_{l_1}^{m_1}(\Omega_1) Y_l^{m*}(\Omega) r^{l+2k} \, \mathrm{d}\boldsymbol{r} \, \mathrm{d}\boldsymbol{r}_1. \tag{8}$$

In order to compute the above integral, we note that the angular integrations must lead to an m-independent answer, since otherwise, when we choose f_{lm} to be independent of m (i.e. $f(\boldsymbol{r})$ independent of ϕ), then $J_{l+2k}(h_{lm})$, which must be independent of m, will not be so. This remark allows us to compute the integral by suitably taking m in the angle-dependent functions. For the present problem, we shall choose $m = l$, the highest possible value. This simplification allows the computation to proceed very systematically. Thus we have

$$J_{l+2k}(h_{lm}) = 4\pi \int f_{lm}(r_1) g(s) Y_l^l(\Omega_1) Y_l^{l*}(\Omega) r^{l+2k} \, \mathrm{d}\boldsymbol{r} \, \mathrm{d}\boldsymbol{r}_1. \tag{9}$$

We use $\boldsymbol{r}_1$ and $\boldsymbol{s}$ as the variables of integration. Taking $\boldsymbol{r}_1 = (r_1, \theta_1, \phi_1)$ and $\boldsymbol{s} = (s, \theta_s, \phi_s)$ as the spherical polar coordinates of $\boldsymbol{r}_1$ and $\boldsymbol{s} = \boldsymbol{r} - \boldsymbol{r}_1$, the spherical polar coordinates (r, θ, ϕ) of $\boldsymbol{r} = \boldsymbol{r}_1 + \boldsymbol{s}$ are related to those of $\boldsymbol{r}_1$, $\boldsymbol{s}$ through the following equations:

$$r \sin\theta \, \mathrm{e}^{\mp \mathrm{i}\phi} = r_1 \sin\theta_1 \, \mathrm{e}^{\mp \mathrm{i}\phi_1} + s \sin\theta_s \, \mathrm{e}^{\mp \mathrm{i}\phi_s} \tag{10}$$

$$r^2 = r_1^2 + s^2 + 2r_1 s[\cos\theta \cos\theta_s + \sin\theta \sin\theta_s \cos(\phi_1 - \phi_s)]. \tag{11}$$

Also (see Edmonds 1960 and equation (2.5.5), p 21 of this book)

$$Y_l^l(\Omega) = (-1)^l Y_l^{l*}(\Omega) = \frac{(-1)^l}{2^l l!}\left(\frac{(2l+1)!}{4\pi}\right)^{1/2} (\sin\theta \, \mathrm{e}^{\mathrm{i}\phi})^l. \tag{12}$$

In other words,

$$Y_l^l(\Omega_1) Y_l^{l*}(\Omega) r^{l+2k}$$

$$= \frac{(2l+1)!}{4\pi \,.\, 2^{2l} \,.\, (l!)^2} (\sin\theta_1 \, \mathrm{e}^{\mathrm{i}\phi_1})^l (r \, \mathrm{e}^{-\mathrm{i}\phi} \sin\theta)^l$$

$$\times \{r_1^2 + s^2 + 2r_1 s[\cos\theta_1 \cos\theta_s + \sin\theta_1 \sin\theta_s \cos(\phi_1 - \phi_s)]\}^k$$

the right-hand side of which becomes, on using equation (10),

$$\frac{(2l+1)!}{4\pi \,.\, 2^{2l} \,.\, (l!)^2} l!\,k! \sum_{p,q,t} \frac{2^t}{p!(l-p)!t!q!(k-q-t)!} (r_1)^{l+2k-p-2q-t} (s)^{p+2q+t}$$

$$\times [\cos\theta_1 \cos\theta_s + \sin\theta_1 \sin\theta_s \cos(\phi_1 - \phi_s)]^t (\sin\theta_1)^{2l-p} (\sin\theta_s)^p \, \mathrm{e}^{\mathrm{i}p(\phi_1 - \phi_s)}.$$

Substituting from the above in equation (9), we find that the angular integral we require is

$$2^t \int [\cos\theta_1 \cos\theta_s + \sin\theta_1 \sin\theta_s \cos(\phi_1 - \phi_s)]^t (\sin\theta_1)^{2l-p+1} (\sin\theta_s)^{p+1} \, \mathrm{e}^{\mathrm{i}p(\phi_1 - \phi_s)}$$

$$\times \mathrm{d}\theta_1 \, \mathrm{d}\phi_1 \, \mathrm{d}\theta_s \, \mathrm{d}\phi_s$$

which is $4\pi^3 l!t!/\Gamma(l+\frac{3}{2})(\frac{1}{2}(t-p))!\Gamma(\frac{1}{2}(t+p)+\frac{3}{2})$ whenever $t\mp p$ are even, non-negative integers and zero otherwise.

Thus we arrive at†

$$J_{l+2k}(h_{lm}) = \frac{4\pi^3(2l+1)!k!}{2^{2l}\Gamma(l+\frac{3}{2})} \times \sum_{p,q,t} \frac{\int dr_1\, ds f_{lm}(r_1) g(s) r_1^{l+2k-p-2q-t+2} s^{p+2q+t+2}}{p!(l-p)!q!(k-q-t)!(\frac{1}{2}(t-p))!\Gamma(\frac{1}{2}(t+p)+\frac{3}{2})} \tag{13}$$

where $t\mp p$ are even non-negative integers.

Replacing the summation index t by $2t-2q-p$ where the new t will now be a non-negative integer, equation (13) becomes, using the definition in equation (5),

$$J_{l+2k}(h_{lm}) = \frac{\pi(2l+1)!k!}{2^{2l+2}\Gamma(l+\frac{3}{2})} \times \sum_{t,p,q} \frac{J_{l+2k-2t}(f_{lm}) J_{2t}(g)}{p!q!(l-p)!(k-2t+p+q)!(t-p-q)!\Gamma(t-q+\frac{3}{2})}. \tag{14}$$

The p, q summations can be immediately performed using equation (A1.1) of Edmonds (1960). Then an application of the duplication formula $\Gamma(2z) = (2^{2z-1}/\surd\pi)\Gamma(z)\Gamma(z+\frac{1}{2})$ for the gamma functions yields the final answer

$$J_{l+2k}(h_{lm}) = \tfrac{1}{2}\surd(\pi) k!\Gamma(k+l+\tfrac{3}{2}) \sum_t \frac{J_{l+2k-2t}(f_{lm}) J_{2t}(g)}{t!(k-t)!\Gamma(t+\frac{3}{2})\Gamma(k+l-t+\frac{3}{2})} \tag{15}$$

from which the special cases given in Satchler (1972) can be immediately written down.

In another paper under preparation, we shall present results for the case where the spherically symmetric function g is replaced by a multipole $\surd(4\pi) g_{lm}(r) Y_l^m(\Omega)$.

References

Edmonds A R 1960 *Angular Momentum in Quantum Mechanics* (Princeton N J: Princeton University Press)
Satchler G R 1972 *J. Math. Phys.* **13** 1118

† The ranges of all summations in the paper are dictated by the non-negativity of the arguments for all the factorials present. In such a situation, the ranges are usually omitted.